

# Eigenvalues Associated with Prolate Spheroidal Wave Functions of Zero Order

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*Presented here are tables of values of  $\chi_n$  and  $\lambda_n$ , quantities defined by the eigenvalue problems*

$$(1 - x^2)\psi_n'' - 2x\psi_n' + (\chi_n - c^2x^2)\psi_n = 0$$

*and*

$$\lambda_n\psi_n(x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} \psi_n(y)dy.$$

*In addition, some approximations for these quantities are given and evaluated.*

The prolate spheroidal wave functions of zero order,  $\psi_n(x)$ ,  $n = 0, 1, \dots$ , are bounded continuous solutions of both the differential equation

$$(1 - x^2) \frac{d^2\psi_n}{dx^2} - 2x \frac{d\psi_n}{dx} + (\chi_n - c^2x^2)\psi_n = 0$$

and the integral equation

$$\lambda_n\psi_n(x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} \psi_n(y)dy.$$

The importance of these functions and the corresponding eigenvalues  $\chi_n$  and  $\lambda_n$  for a great variety of problems, dealing with such diverse matters as lasers, communication theory, optics, noise theory, etc., can be found in the bibliographies of Refs. 1 and 2. It is our purpose here in response to numerous requests to present some numerical values for these eigenvalues.

Tables I and II list values of  $\chi_n$  and  $\lambda_n$  respectively for  $n = 0(1)20(5)40$  and  $c = 0(1)20(5)40$ . The values given are, we believe, accurate to all eight figures listed.\* Results of the computation are shown graphically on Figs. 1-4.

The values of  $\chi_n$  were obtained using the method of Bouwkamp as explained for example in Flammer.<sup>1</sup> This computation also gives expansion coefficients  $d_r^{on}(c)$  in Flammer's notation, from which his quantity  $R_{on}^{(1)}(c, 1)$  can be computed. The  $\lambda$ 's were then found from

$$\lambda_n = \frac{2c}{\pi} [R_{on}^{(1)}(c, 1)]^2.$$

The tables presented required 0.027 hours of computing time on the IBM 7090.

The following formulae for  $\lambda_n$  and  $\chi_n$  are given in Ref. 2. For fixed  $n$  and small  $c$

$$\lambda_n = \frac{2}{\pi} \left[ \frac{2^{2n}(n!)^3}{(2n)!(2n+1)!} \right]^2 c^{2n+1} \cdot \left[ 1 - \frac{(2n+1)c^2}{(2n-1)^2(2n+3)^2} + o(c^4) \right]. \quad (1)$$

For fixed  $n$  and large  $c$

$$1 - \lambda_n = \frac{2^{3n+2} \sqrt{\pi} c^{n+\frac{1}{2}} e^{-2c}}{n!} \left[ 1 - \frac{6n^2 - 2n + 3}{32c} + o\left(\frac{1}{c^2}\right) \right]. \quad (2)$$

Some values computed from the terms explicitly exhibited in (1) and (2) are shown as dotted lines on Figs. 3 and 4.

For  $n$  and  $c$  both large, we have the following result. Let  $b$  be fixed and let

$$n = \left[ \frac{2}{\pi} (c + b \ln 2\sqrt{c}) \right] \quad (3)$$

where the brackets denote "integer part of". Then

$$\lim_{c \rightarrow \infty} \lambda_n = (1 + e^{\pi b})^{-1}. \quad (4)$$

\* The notation  $E \pm XY$  following an entry in the tables indicates that the entry is to be multiplied by  $10^{\pm XY}$  where  $XY$  is an integer in decimal notation, e.g.,  $E + 03$  denotes a factor of  $10^3$ .

TABLE I— $\chi_n$ 

$c =$	1.	2.	3.	4.	5.	6.
$n$						
0	3.1900006E - 01	1.1277341E + 00	2.1367322E + 00	3.1720674E + 00	4.1951289E + 00	5.2082692E + 00
1	2.5930846E + 00	4.2871285E + 00	6.8208883E + 00	9.8059438E + 00	1.2911703E + 01	1.6000443E + 01
2	6.5334718E + 00	8.2257130E + 00	1.1192939E + 01	1.5306300E + 01	2.0176915E + 01	2.5356479E + 01
3	1.2514462E + 01	1.4100204E + 01	1.6889030E + 01	2.1048961E + 01	2.6587360E + 01	3.3204199E + 01
4	2.0508274E + 01	2.2054830E + 01	2.4708535E + 01	2.8596855E + 01	3.3897096E + 01	4.0720194E + 01
5	3.0503405E + 01	3.2035263E + 01	3.4631281E + 01	3.8367138E + 01	4.3358996E + 01	4.9773712E + 01
6	4.2503818E + 01	4.4024748E + 01	4.6591428E + 01	5.0252698E + 01	5.5080962E + 01	6.1180757E + 01
7	5.6502845E + 01	5.8018371E + 01	6.0567636E + 01	6.4186116E + 01	6.8924773E + 01	7.4852867E + 01
8	7.2502203E + 01	7.4014194E + 01	7.6552160E + 01	8.0143235E + 01	8.4825931E + 01	9.0651159E + 01
9	9.0501757E + 01	9.2011304E + 01	9.4541490E + 01	9.8113806E + 01	1.0275858E + 02	1.0851545E + 02
10	1.1050143E + 02	1.1200922E + 02	1.1453381E + 02	1.1809267E + 02	1.2271039E + 02	1.2841888E + 02
11	1.3250119E + 02	1.3400766E + 02	1.3652809E + 02	1.4007696E + 02	1.4467463E + 02	1.5034744E + 02
12	1.5650101E + 02	1.5800647E + 02	1.6052372E + 02	1.6406496E + 02	1.6864733E + 02	1.7429300E + 02
13	1.8250086E + 02	1.8400554E + 02	1.8652029E + 02	1.9005557E + 02	1.9462601E + 02	2.0025051E + 02
14	2.1050075E + 02	2.1200480E + 02	2.1451756E + 02	2.1804808E + 02	2.2260902E + 02	2.2821669E + 02
15	2.4050065E + 02	2.4200419E + 02	2.4451535E + 02	2.4804202E + 02	2.5259526E + 02	2.5818931E + 02
16	2.7250058E + 02	2.7400370E + 02	2.7651353E + 02	2.8003704E + 02	2.8458396E + 02	2.9016684E + 02
17	3.0650051E + 02	3.0800328E + 02	3.1051202E + 02	3.1403289E + 02	3.1857456E + 02	3.2414815E + 02
18	3.4250046E + 02	3.4400294E + 02	3.4651075E + 02	3.5002941E + 02	3.5456666E + 02	3.6013245E + 02
19	3.8050041E + 02	3.8200264E + 02	3.8450967E + 02	3.8802645E + 02	3.9255596E + 02	3.9811912E + 02
20	4.2050037E + 02	4.2200239E + 02	4.2450874E + 02	4.2802392E + 02	4.3255422E + 02	4.3810771E + 02
25	6.5050024E + 02	6.5200154E + 02	6.5450564E + 02	6.5801543E + 02	6.6253497E + 02	6.6809946E + 02
30	9.3050017E + 02	9.3200108E + 02	9.3450394E + 02	9.3801078E + 02	9.4252442E + 02	9.4804850E + 02
35	1.2605001E + 03	1.2620008E + 03	1.2645029E + 03	1.2680079E + 03	1.2725180E + 03	1.2780358E + 03
40	1.6405001E + 03	1.6420006E + 03	1.6445022E + 03	1.6480061E + 03	1.6525138E + 03	1.6580275E + 03

TABLE I—Continued

$\epsilon =$ $n$	7.	8.	9.	10.	11.	12.
0	6.2162529E + 00	7.2215789E + 00	8.2254064E + 00	9.2283043E + 00	1.0230581E + 01	1.1232421E + 01
1	1.9056678E + 01	2.2092154E + 01	2.5116120E + 01	2.8133464E + 01	3.1146682E + 01	3.4157135E + 01
2	3.0560201E + 01	3.5706417E + 01	4.0802950E + 01	4.5868953E + 01	5.0916879E + 01	5.5953514E + 01
3	4.0405727E + 01	4.7757099E + 01	5.5051178E + 01	6.2257700E + 01	6.9401323E + 01	7.6605824E + 01
4	4.8910585E + 01	5.8016770E + 01	6.7500818E + 01	7.6993289E + 01	8.6369707E + 01	9.5638659E + 01
5	5.7777751E + 01	6.7364750E + 01	7.8205025E + 01	8.9739267E + 01	1.0144734E + 02	1.1305411E + 02
6	6.8701439E + 01	7.7825223E + 01	8.8658000E + 01	1.0103543E + 02	1.1446976E + 02	1.2835139E + 02
7	8.2064637E + 01	9.0691430E + 01	1.0090790E + 02	1.1288107E + 02	1.2660565E + 02	1.4174147E + 02
8	9.7684571E + 01	1.0601169E + 02	1.1574796E + 02	1.2705083E + 02	1.4010628E + 02	1.5502454E + 02
9	1.1543428E + 02	1.2357716E + 02	1.3302232E + 02	1.4387201E + 02	1.5626473E + 02	1.7038033E + 02
10	1.3525788E + 02	1.4327579E + 02	1.5253134E + 02	1.6309665E + 02	1.7506323E + 02	1.8855245E + 02
11	1.5712799E + 02	1.6505554E + 02	1.7417688E + 02	1.8454762E + 02	1.9623470E + 02	2.0932095E + 02
12	1.8102925E + 02	1.8888879E + 02	1.9791014E + 02	2.0813839E + 02	2.1962634E + 02	2.3243647E + 02
13	2.0695232E + 02	2.1475915E + 02	2.2370349E + 02	2.3382285E + 02	2.4516097E + 02	2.5776775E + 02
14	2.3489114E + 02	2.4265622E + 02	2.5153975E + 02	2.6137378E + 02	2.7279496E + 02	2.8524510E + 02
15	2.6484166E + 02	2.7257305E + 02	2.8140764E + 02	2.9137313E + 02	3.0250101E + 02	3.1482692E + 02
16	2.9680105E + 02	3.0450484E + 02	3.1329940E + 02	3.2320865E + 02	3.3426093E + 02	3.4648622E + 02
17	3.3076730E + 02	3.3844819E + 02	3.4720956E + 02	3.5707281E + 02	3.6806210E + 02	3.8020452E + 02
18	3.6673895E + 02	3.7440061E + 02	3.8313413E + 02	3.9295859E + 02	4.0389544E + 02	4.1596869E + 02
19	4.0471489E + 02	4.1236024E + 02	4.2107018E + 02	4.3086179E + 02	4.4175430E + 02	4.5376914E + 02
20	4.4469430E + 02	4.5232571E + 02	4.6101548E + 02	4.7077902E + 02	4.8163367E + 02	4.9359870E + 02
25	6.7462529E + 02	6.8220999E + 02	6.9083226E + 02	7.0050200E + 02	7.1123026E + 02	7.2302932E + 02
30	9.5458747E + 02	9.6214660E + 02	9.7073194E + 02	9.8035039E + 02	9.9100965E + 02	1.0027182E + 03
35	1.2845645E + 03	1.2921081E + 03	1.3006711E + 03	1.3102584E + 03	1.3208759E + 03	1.3325297E + 03
40	1.6645496E + 03	1.6720830E + 03	1.6806314E + 03	1.6901985E + 03	1.7007886E + 03	1.7124067E + 03

TABLE I—Continued

$c =$	13.	14.	15.	16.	17.	18.
$n$						
0	1.2233939E + 01	1.3235214E + 01	1.4236300E + 01	1.5237237E + 01	1.6238054E + 01	1.7238772E + 01
1	3.7165631E + 01	4.0172681E + 01	4.3178630E + 01	4.6183721E + 01	4.9188129E + 01	5.2191983E + 01
2	6.0982596E + 01	6.6006328E + 01	7.1026104E + 01	7.6042858E + 01	8.1057244E + 01	8.6089739E + 01
3	8.3585719E + 01	9.0649230E + 01	9.7701181E + 01	1.0474459E + 02	1.1178148E + 02	1.1881324E + 02
4	1.0483494E + 02	1.1398465E + 02	1.2310348E + 02	1.3220071E + 02	1.4128205E + 02	1.5035127E + 02
5	1.2450713E + 02	1.3584050E + 02	1.4709398E + 02	1.5829441E + 02	1.6945806E + 02	1.8059490E + 02
6	1.4223009E + 02	1.5593166E + 02	1.6945804E + 02	1.8285785E + 02	1.9617229E + 02	2.0942797E + 02
7	1.5768187E + 02	1.7382917E + 02	1.8983663E + 02	2.0562050E + 02	2.2121898E + 02	2.3668721E + 02
8	1.7164637E + 02	1.8946449E + 02	2.0780930E + 02	2.2615098E + 02	2.4425790E + 02	2.6481922E + 02
9	1.8639078E + 02	2.0430123E + 02	2.2377440E + 02	2.4417004E + 02	2.6481922E + 02	2.8529987E + 02
10	2.0372711E + 02	2.2077813E + 02	2.3982549E + 02	2.6072449E + 02	2.8296138E + 02	3.0581686E + 02
11	2.2391335E + 02	2.4015533E + 02	2.5823139E + 02	2.7832097E + 02	3.0044546E + 02	3.2428066E + 02
12	2.4664450E + 02	2.6234574E + 02	2.7966575E + 02	2.9877161E + 02	3.1985976E + 02	3.4306306E + 02
13	2.7170195E + 02	2.8703335E + 02	3.0384785E + 02	3.2225590E + 02	3.4240470E + 02	3.6448308E + 02
14	2.9897207E + 02	3.1403113E + 02	3.3048712E + 02	3.4841822E + 02	3.6792260E + 02	3.8912905E + 02
15	3.2839115E + 02	3.4323943E + 02	3.5942402E + 02	3.7700550E + 02	3.9605573E + 02	4.1666293E + 02
16	3.5991943E + 02	3.7459944E + 02	3.9056997E + 02	4.096376E + 02	4.2688830E + 02	4.4675937E + 02
17	3.9353027E + 02	4.0807299E + 02	4.2387018E + 02	4.4096376E + 02	4.6942089E + 02	4.7923537E + 02
18	4.2920498E + 02	4.4363385E + 02	4.5928798E + 02	4.7620357E + 02	4.9442089E + 02	5.1398496E + 02
19	4.6693004E + 02	4.8126322E + 02	4.9679753E + 02	5.1356471E + 02	5.3159977E + 02	5.5094143E + 02
20	5.0669542E + 02	5.2094728E + 02	5.3638001E + 02	5.5302178E + 02	5.7090346E + 02	5.9005891E + 02
25	7.3591267E + 02	7.4989504E + 02	7.6499245E + 02	7.8122221E + 02	7.9860305E + 02	8.1715510E + 02
30	1.0154855E + 03	1.0293217E + 03	1.0423378E + 03	1.0602457E + 03	1.0773582E + 03	1.0955890E + 03
35	1.3452287E + 03	1.3589745E + 03	1.3737811E + 03	1.3899552E + 03	1.4066063E + 03	1.4246442E + 03
40	1.7250580E + 03	1.7387482E + 03	1.7534835E + 03	1.7692707E + 03	1.7861108E + 03	1.8040295E + 03

TABLE I—Continued

$\epsilon =$ #	19.	20.	25.	30.	35.	40.
0	1.8239408E + 01	1.9239976E + 01	2.4242094E + 01	2.9243472E + 01	3.4244440E + 01	3.92445159E + 01
1	5.5195383E + 01	5.8198404E + 01	7.3209570E + 01	8.8216755E + 01	1.0322177E + 02	1.1822547E + 02
2	9.1080693E + 01	9.6090388E + 01	1.2112584E + 02	1.4614836E + 02	1.7116395E + 02	1.9617538E + 02
3	1.2584090E + 02	1.3286522E + 02	1.6795309E + 02	2.0300813E + 02	2.3804589E + 02	2.7307342E + 02
4	1.5941100E + 02	1.6846310E + 02	2.1364862E + 02	2.5876280E + 02	3.0384036E + 02	3.4889654E + 02
5	1.9171143E + 02	2.0281205E + 02	2.5816358E + 02	3.1337546E + 02	3.6851770E + 02	4.2361994E + 02
6	2.2641133E + 02	2.3582286E + 02	3.0144041E + 02	3.6680477E + 02	4.3204536E + 02	4.9721681E + 02
7	2.5206574E + 02	2.6738042E + 02	3.4341554E + 02	4.1900403E + 02	4.9438749E + 02	5.6965810E + 02
8	2.7978768E + 02	2.9732623E + 02	3.8400598E + 02	4.6991994E + 02	5.5550429E + 02	6.4091212E + 02
9	3.0548786E + 02	3.2541914E + 02	4.2311389E + 02	5.1949088E + 02	6.1555127E + 02	7.1094418E + 02
10	3.2868309E + 02	3.5126388E + 02	4.6061231E + 02	5.6764441E + 02	6.7387820E + 02	7.7971605E + 02
11	3.4916478E + 02	3.7436419E + 02	4.9632769E + 02	6.1429371E + 02	7.3102784E + 02	8.4718535E + 02
12	3.6825795E + 02	3.9493519E + 02	5.2999995E + 02	6.5933191E + 02	7.8673407E + 02	9.1330472E + 02
13	3.8867917E + 02	4.1503422E + 02	5.6120901E + 02	7.0262235E + 02	8.4091938E + 02	9.7802077E + 02
14	4.1220822E + 02	4.3736223E + 02	5.8939527E + 02	7.4397983E + 02	8.9349109E + 02	1.0412727E + 03
15	4.3894029E + 02	4.6303801E + 02	6.1457579E + 02	7.8313074E + 02	9.4433551E + 02	1.1029905E + 03
16	4.6847456E + 02	4.9183371E + 02	6.3866220E + 02	8.1963882E + 02	9.930804E + 02	1.1630922E + 03
17	5.0052914E + 02	5.2335572E + 02	6.6469676E + 02	8.5289508E + 02	1.0402144E + 03	1.2214800E + 03
18	5.3494661E + 02	5.5736402E + 02	6.9431366E + 02	8.8272216E + 02	1.0847715E + 03	1.2780344E + 03
19	5.7163273E + 02	5.9372195E + 02	7.2743245E + 02	9.1079959E + 02	1.1265337E + 03	1.3326043E + 03
20	6.1052541E + 02	6.3234422E + 02	7.6357631E + 02	9.4039863E + 02	1.1648560E + 03	1.3849884E + 03
25	8.3690003E + 02	8.5786114E + 02	9.8185790E + 02	1.1410460E + 03	1.3423387E + 03	1.6007929E + 03
30	1.1149527E + 03	1.1354649E + 03	1.2558699E + 03	1.4079656E + 03	1.5952330E + 03	1.8229611E + 03
35	1.4437796E + 03	1.4640237E + 03	1.5823180E + 03	1.7304322E + 03	1.9106676E + 03	2.1260735E + 03
40	1.8230168E + 03	1.8430874E + 03	1.9600239E + 03	2.1056213E + 03	2.2815621E + 03	2.4899642E + 03

TABLE II— $\lambda_n$ 

$n$	$c =$	1.	2.	3.	4.	5.	6.
0		5.7258178E	— 01	9.7582863E	— 01	9.9935241E	— 01
1		6.2791274E	— 02	7.0996324E	— 01	9.7986456E	— 01
2		1.2374793E	— 03	2.0513868E	— 01	7.9992193E	— 01
3		9.2009770E	— 06	1.8203800E	— 02	3.4356219E	— 01
4		3.7179280E	— 08	7.0814710E	— 04	5.6015551E	— 02
5		9.4914367E	— 11	1.6551244E	— 05	4.1820948E	— 03
6		1.6715716E	— 13	2.6410165E	— 07	1.9330846E	— 04
7		2.1544491E	— 16	3.0737365E	— 09	6.3991502E	— 06
8		2.1207239E	— 19	2.7281307E	— 11	1.5822998E	— 07
9		1.6466214E	— 22	1.9085689E	— 13	3.0917257E	— 09
10		1.0343492E	— 25	1.0797906E	— 15	4.8757393E	— 11
11		5.3650197E	— 29	5.0431156E	— 18	6.3402794E	— 13
12		2.3367231E	— 32	1.9775436E	— 20	6.9173022E	— 15
13		8.6674831E	— 36	6.6033063E	— 23	6.4235507E	— 17
14		2.7709612E	— 39	1.9002929E	— 25	5.1393068E	— 19
15	0.		1.4873466E	4.7620029E	— 28	3.5797463E	— 21
16	0.		1.6563614E	1.0485031E	— 30	2.1905130E	— 23
17	0.		1.6207613E	2.0444867E	— 33	1.869344E	— 25
18	0.		0.	3.5551880E	— 36	5.7350388E	— 28
19	0.		0.	5.5475853E	— 39	2.4864675E	— 30
20	0.		0.	0.		9.7273155E	— 33
25	0.		0.	0.		0.	0.
30	0.		0.	0.		0.	0.
35	0.		0.	0.		0.	0.
40	0.		0.	0.		0.	0.

TABLE II—Continued

$c =$ $n$	7.	8.	9.	10.	11.	12.
0	9.9998546E-01	9.9999787E-01	9.9999969E-01	9.9999996E-01	9.9999999E-01	1.0000000E+00
1	9.9929217E-01	9.9987898E-01	9.9997999E-01	9.9999677E-01	9.9999949E-01	9.9999992E-01
2	9.9870806E-01	9.9700462E-01	9.9941873E-01	9.9989273E-01	9.9989091E-01	9.9999670E-01
3	8.6456615E-01	9.6054568E-01	9.9039622E-01	9.9790124E-01	9.9957158E-01	9.9991663E-01
4	4.7705272E-01	7.4790284E-01	9.1013316E-01	9.7445778E-01	9.9371700E-01	9.9858732E-01
5	1.1572386E-01	3.2027663E-01	5.9909617E-01	8.2514635E-01	9.4136927E-01	9.8366430E-01
6	1.3055972E-02	6.0784427E-02	1.9693935E-01	4.4015011E-01	7.0394130E-01	8.8175663E-01
7	9.0657300E-04	6.1262894E-03	3.0565075E-02	1.1232482E-01	2.9607849E-01	5.5736081E-01
8	4.5623948E-05	4.1825206E-04	2.8466070E-03	1.4920175E-02	6.0370339E-02	1.8342927E-01
9	1.7774751E-06	2.1663088E-05	1.9230822E-04	1.3145890E-03	7.1417030E-03	3.1054179E-02
10	5.5226131E-08	8.9304272E-07	1.0194316E-05	8.8213430E-05	6.0469421E-04	3.3745471E-03
11	1.4251398E-09	3.0137350E-08	4.3973999E-07	4.7664454E-06	4.0395675E-05	2.7741888E-04
12	3.0622379E-11	8.4965846E-10	1.5795600E-08	2.1339628E-07	2.2179166E-06	1.8475085E-05
13	5.9228434E-13	2.0334083E-11	4.8068821E-10	8.0707164E-09	1.0243298E-07	1.0282524E-06
14	8.7926605E-15	4.1852675E-13	1.2564804E-11	2.6170188E-10	4.0455355E-09	4.8758791E-08
15	1.2026890E-16	7.4905020E-15	2.8533973E-13	7.3634903E-12	1.3840557E-10	1.9981456E-09
16	1.4445726E-18	1.1767148E-16	5.6843266E-15	1.8159383E-13	4.1453619E-12	7.1571886E-11
17	1.5359357E-20	1.6358709E-18	1.0016999E-16	3.9589753E-15	1.0966649E-13	2.2619074E-12
18	1.4559023E-22	2.0270123E-20	1.5727550E-18	7.6870812E-17	2.5823710E-15	6.3575326E-14
19	1.2380854E-24	2.2529462E-22	2.2145250E-20	1.3380681E-18	5.4488496E-17	1.6002320E-15
20	9.4989023E-27	2.2588880E-24	2.8123556E-22	2.1001719E-20	1.0363386E-18	3.6290304E-17
25	6.341093E-38	5.7412040E-35	2.3265268E-32	4.9987005E-30	6.4253487E-28	5.4015219E-26
30	0.	0.	0.	0.	0.	1.1037482E-35
35	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.



TABLE II — Continued

$c =$ $n$	13.	14.	15.	16.	17.	18.
0	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00
1	9.9999999E - 01	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00
2	9.9999944E - 01	9.9999991E - 01	9.9999998E - 01	1.0000000E + 00	1.0000000E + 00	1.0000000E + 00
3	9.9998436E - 01	9.999715E - 01	9.9999949E - 01	9.9999991E - 01	1.0000000E + 00	1.0000000E + 00
4	9.9970100E - 01	9.9993948E - 01	9.9998818E - 01	9.999776E - 01	9.9999958E - 01	9.9999992E - 01
5	9.9595266E - 01	9.9907074E - 01	9.9979787E - 01	9.9995783E - 01	9.9999149E - 01	9.9999833E - 01
6	9.6225505E - 01	9.8963945E - 01	9.9741834E - 01	9.9939756E - 01	9.9986611E - 01	9.9997138E - 01
7	7.8874478E - 01	9.2170099E - 01	9.7594492E - 01	9.9346756E - 01	9.9830416E - 01	9.9961326E - 01
8	4.0633176E - 01	6.6365081E - 01	8.5371077E - 01	9.4900699E - 01	9.8478345E - 01	9.9589845E - 01
9	1.0588399E - 01	2.7254759E - 01	5.1899118E - 01	7.5367260E - 01	9.0120813E - 01	9.6721066E - 01
10	1.5487208E - 02	5.7772021E - 02	1.6922485E - 01	3.7484512E - 01	6.2548476E - 01	8.2543081E - 01
11	1.5807741E - 03	7.5604029E - 03	3.0214721E - 02	9.8343344E - 02	2.5013211E - 01	4.8298471E - 01
12	1.2703997E - 04	7.3609236E - 04	3.6365712E - 03	1.5328905E - 02	5.4203716E - 02	1.5521339E - 01
13	8.4414630E - 06	5.8099164E - 05	3.4130591E - 04	1.7310585E - 03	7.6061834E - 03	2.8698205E - 02
14	4.7534082E - 07	3.8540665E - 06	2.6544166E - 05	1.5775571E - 04	8.1779798E - 04	3.7164510E - 03
15	2.3057257E - 08	2.1923715E - 07	1.7585578E - 06	1.2118149E - 05	7.2740820E - 05	3.8416435E - 04
16	9.7549162E - 10	1.0846150E - 08	1.0092888E - 07	8.0200884E - 07	5.5289258E - 06	3.3476998E - 05
17	3.6389074E - 11	4.7180556E - 10	5.0802489E - 09	4.6393027E - 08	3.6561510E - 07	2.5213492E - 06
18	1.2039857E - 12	1.8208273E - 11	2.2646216E - 10	2.3711707E - 09	2.1297926E - 08	1.6661751E - 07
19	3.5675475E - 14	6.2816688E - 13	9.0124935E - 12	1.0801803E - 10	1.1034773E - 09	9.7699046E - 09
20	5.186419E - 16	1.9498859E - 14	3.2240722E - 13	4.4178809E - 12	5.1252279E - 11	5.1225654E - 10
25	3.1773164E - 24	1.3781710E - 22	4.5947254E - 21	1.2173462E - 19	2.6337453E - 18	4.7585812E - 17
30	1.4501905E - 33	1.3251429E - 31	8.8515118E - 30	4.4989186E - 28	1.7978779E - 26	5.8040997E - 25
35	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.

TABLE II—Continued

$c =$ $n$	19.	20.	25.	30.	35.	40.
0	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
1	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
2	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
3	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
4	9.999999E-01	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
5	9.999998E-01	9.999994E-01	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
6	9.999408E-01	9.999881E-01	1.000000E+00	1.000000E+00	1.000000E+00	1.000000E+00
7	9.999125E-01	9.999809E-01	9.999999E-01	1.000000E+00	1.000000E+00	1.000000E+00
8	9.989683E-01	9.997534E-01	9.999998E-01	1.000000E+00	1.000000E+00	1.000000E+00
9	9.904264E-01	9.974325E-01	9.999821E-01	1.000000E+00	1.000000E+00	1.000000E+00
10	9.3461880E-01	9.7911569E-01	9.9997682E-01	9.999999E-01	1.000000E+00	1.000000E+00
11	7.1923718E-01	8.7971361E-01	9.9974565E-01	9.9999983E-01	1.000000E+00	1.000000E+00
12	3.4534703E-01	5.8879338E-01	9.9766185E-01	9.9999783E-01	1.000000E+00	1.000000E+00
13	9.0528307E-02	2.2898871E-01	9.8251216E-01	9.997547E-01	9.9999998E-01	1.000000E+00
14	1.4751648E-02	5.0245996E-02	9.0214476E-01	9.9975907E-01	9.9999980E-01	1.000000E+00
15	1.7952937E-03	7.4212338E-03	6.5129574E-01	9.9796698E-01	9.9999766E-01	1.000000E+00
16	1.7967267E-04	8.5983868E-04	2.9167771E-01	9.8564508E-01	9.9997604E-01	1.000000E+00
17	1.5383334E-05	8.3739541E-05	7.5468799E-02	9.2101083E-01	9.9978298E-01	9.9999978E-01
18	1.1493460E-06	7.0600702E-06	1.3031043E-02	7.0692287E-01	9.9828070E-01	9.9999766E-01
19	7.5908731E-08	5.2374892E-07	1.7588754E-03	3.5647590E-01	9.8836235E-01	9.9997777E-01
20	4.4748828E-09	3.4574493E-08	2.0082884E-04	1.0627740E-01	9.3662832E-01	9.9981076E-01
25	7.3154280E-16	6.7594641E-15	6.7594641E-10	4.7379571E-06	5.3273087E-03	4.8731168E-01
30	1.5483211E-23	3.4788064E-22	2.4882077E-16	1.3177338E-11	1.0489720E-07	1.7876070E-04
35	6.1447670E-32	2.3202540E-30	1.6231560E-23	5.8407336E-18	2.5646331E-13	2.2391506E-09
40	0.	0.	2.4836333E-31	5.8205690E-25	1.3022033E-19	4.9862021E-15

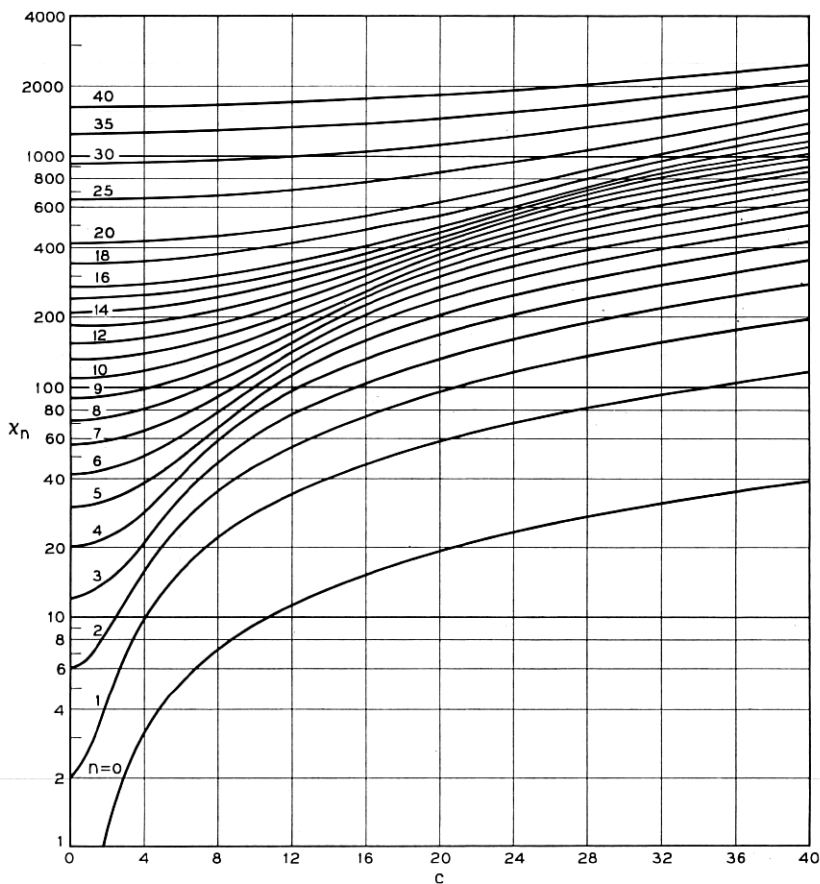


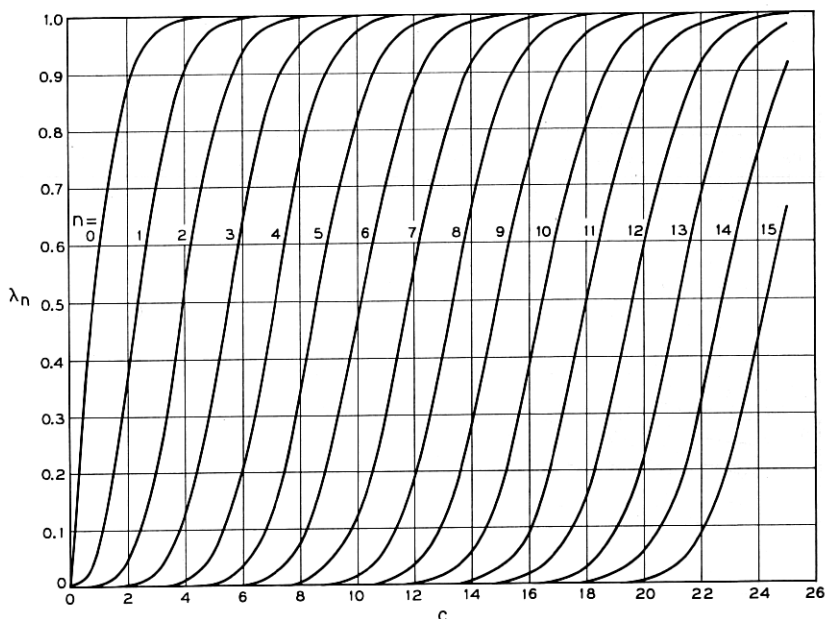
Fig. 1 — Eigenvalues,  $\chi_n$ , of  $(1 - x^2)\psi'' - 2x\psi' + (\chi - c^2x^2)\psi = 0$ .

The derivation of (3) and (4) given in Ref. 2 suggests the approximate formula

$$\lambda_n \approx \hat{\lambda}_n = (1 + e^{\pi \hat{b}})^{-1} \quad (5)$$

$$\hat{b} = \frac{n \frac{\pi}{2} - c + \frac{\pi}{4}}{(\gamma/2) + 2 \ln 2 + \frac{1}{2} \ln c} \quad (6)$$

for the near vertical rise portions of the  $\lambda_n$  curves shown on Fig. 2. Here,  $\gamma = 0.5772156649 \dots$  is the Euler-Mascheroni constant. The remarkable accuracy of this approximation is shown on Fig. 5. Here, for

Fig. 2 — Eigenvalues,  $\lambda_n$ , of integral equation. Linear scale.

fixed values of  $n$  and  $\hat{b}$ , we have determined values of  $c$  from (6) and for these values of  $n$  and  $c$  have plotted  $|\hat{\lambda}_n/\lambda_n - 1|$  vs  $n$ . It is seen that for  $0.2 \leq \hat{\lambda}_n \leq 0.9$ , (5) and (6) give an excellent approximation even for small values of  $n$ .

Corresponding formulae for the  $\chi_n$  follow. For fixed  $n$  and small  $c$

$$\chi_n = n(n+1) + \frac{1}{2} \left[ 1 + \frac{1}{(2n-1)(2n+3)} \right] c^2 + O(c^4)$$

and for fixed  $n$  and large  $c$

$$\chi_n = (2n+1)c - \frac{2n^2 + 2n + 3}{4} - \frac{(2n+1)(n^2 + n + 3)}{16c} + O\left(\frac{1}{c^2}\right).$$

If  $n$  and  $c$  become large according to (3) with  $b$  fixed,

$$\chi_n = c^2 + 2bc + \frac{b^2 - 1}{2} - \frac{b^3 - b}{8c} + O\left(\frac{1}{c^2}\right).$$

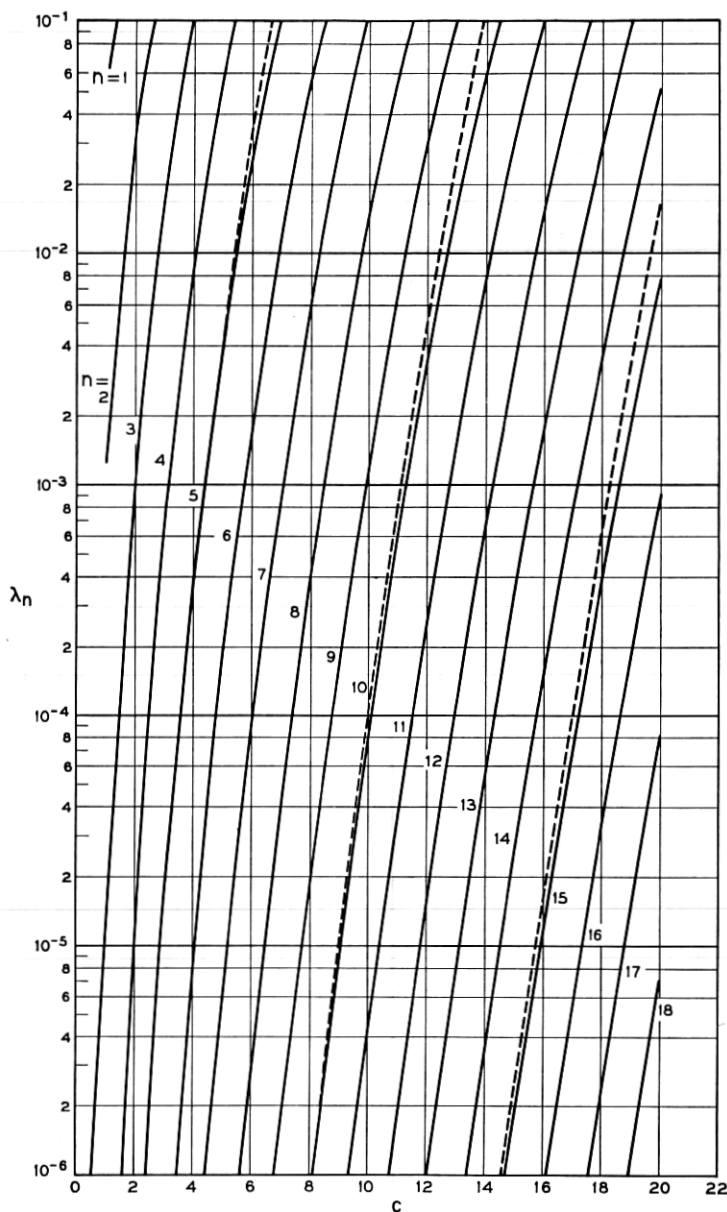


Fig. 3 — Eigenvalues,  $\lambda_n$ , of integral equation for  $c < n\pi/2$ . Dashed lines are approximation (1).

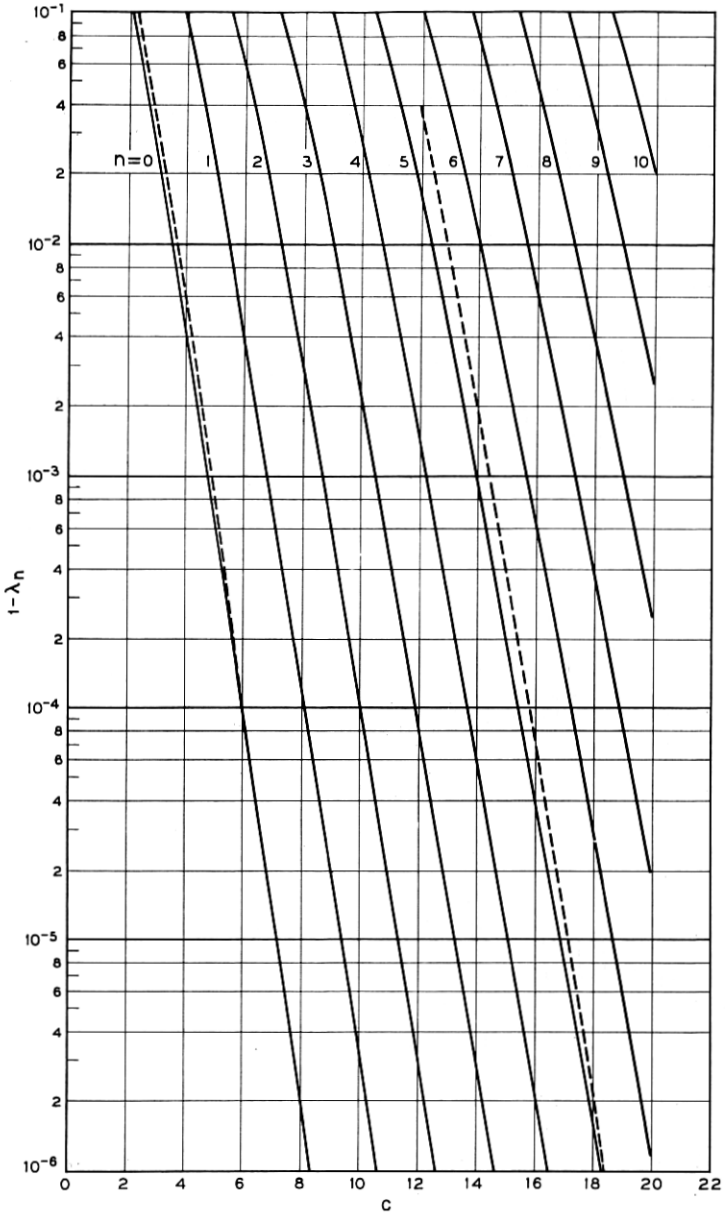


Fig. 4 — Eigenvalues,  $\lambda_n$ , of integral equation for  $c > n\pi/2$ . Dashed lines are approximation (2).

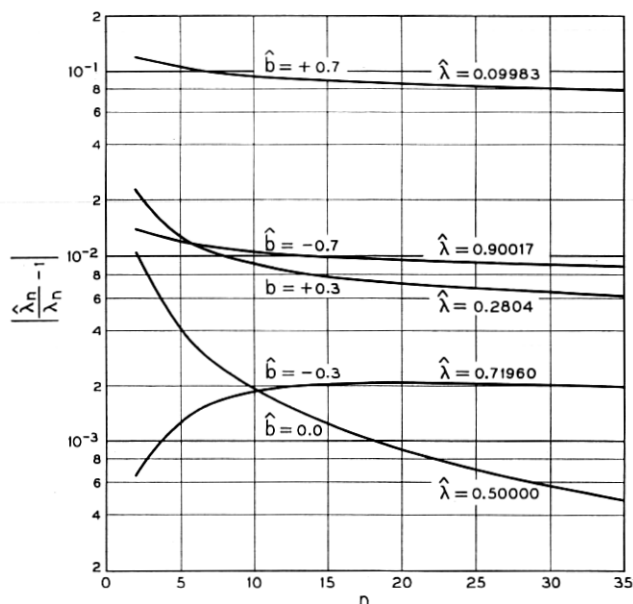


Fig. 5 — Accuracy of the approximation (5) — (6) for the eigenvalues  $\lambda_n$ .

# REFERENCES

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2. Slepian, D., Some Asymptotic Expansions for Prolate Spheroidal Wave Functions, *J. Math. and Phys.*, 44, No. 2, June, 1965, pp. 99-140.

